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Respect pentru oameni și cărți

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Chapter I
PROBLEMS

Notations:

s – semiperimeter of ΔABC, F – area of ΔABC, R – circumradii, r – inradii, h_a, h_b, h_c – altitudes, m_a, m_b, m_c – medians, s_a, s_b, s_c – symedians, w_a, w_b, w_c – internal bisectors, r_a, r_b, r_c – exradii

1. Let a, b, c be positive real numbers. Prove that:

$$\frac{a^5}{a^2+ab+b^2}+\frac{b^5}{b^2+bc+c^2}+\frac{c^5}{c^2+ca+a^2}\geq abc.$$

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2. Prove that in any acute-angled ΔABC with length's sides $a \geq b \geq c$ the following relationship holds:

$$a^3+2b^3+c^3\leq(a+b)(ab+c^2)+(b-c)(bc+a^2).$$

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3. Let a, b, c be positive real numbers with $a+b+c=3$. Prove that:

$$\frac{ab(b+1)}{c}+\frac{bc(c+1)}{a}+\frac{ca(a+1)}{b}\geq 6.$$

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4. In ΔABC; $M, N, P \in [BC]$. Prove that:

$$\sqrt[3]{AM\cdot AN\cdot AP}\left(\frac{1}{AM}+\frac{1}{AN}+\frac{1}{AP}\right)\leq\frac{5}{3}+\frac{2}{3}\left(\frac{b}{c}+\frac{c}{b}\right).$$

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5. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{3}{2}.$$

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6. Prove that if $M \in (BC), N \in (AC), P \in (AB)$ then:

$$\frac{AM^2}{BN+CP}+\frac{BN^2}{CP+AM}+\frac{CP^2}{AM+BN}>\frac{F^2}{2s}\sum\frac{1}{bc\sin^2\frac{A}{2}}.$$

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7. Let a, b, c be positive real numbers. Prove that:

$$\frac{(a^2-ab+b^2)^2}{(a+b)^4}+\frac{(b^2-bc+c^2)^2}{(b+c)^4}+\frac{(c^2-ca+a^2)^2}{(c+a)^4}\geq\frac{3}{16}.$$

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8. Let be $A', A'' \in (BC)$; $B', B'' \in (BC)$; $C', C'' \in (AB)$ in ΔABC; $AA' \cap BB' \cap CC' \neq \emptyset$. Prove that:

$$\frac{27[A'B'C']}{[A''B''C'']} \leq \left(\frac{BA'}{BA''} + \frac{CB'}{CB''} + \frac{AC'}{AC''}\right)^3.$$

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9. Let a, b be positive real numbers with $a^2 + ab + b^2 = k^2, k > 0$. Prove that:

$$\sqrt{a+b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}+1}{\sqrt[4]{3}} \cdot \sqrt{k}.$$

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10. Prove that in any triangle ABC the following relationship holds:

$$3\left(a^a b^b c^c\right)^{\frac{1}{2s}} \geq \sqrt[9]{4RF} \sum \left(a^a b^b c^c\right)^{\frac{1}{3s}}.$$

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11. Let a, b be positive real numbers such that $a^2 + ab + b^2 = 9$. Find the maximal value of expression:

$$(a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 16.$$

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12. Find $x, y, z, t \in \left[0, \frac{\pi}{2}\right]$ such that:

$$\sqrt{\sin x \sin y \sin z \sin t} + \sqrt{\cos x \cos y \cos z \cos t} = 1.$$

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13. Let x, y, z be positive real numbers. Prove that:

$$\sqrt[3]{(2x^3 + 3x^2 + 3x + 1)(2y^3 + 3y^2 + 3y + 1)(2z^3 + 3z^2 + 3z + 1)} \geq$$
$$\geq xyz + 8\sqrt{xyz}.$$

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14. Prove that if $a \leq b \leq c$ in ΔABC then:

$$(m_b m_c)^{m_b - m_c}(m_a m_b)^{m_a - m_b} \geq (m_a m_c)^{m_a - m_c}.$$

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15. Let x, y, z be positive real numbers with $xyz = 1$. Prove that:

$$\frac{\sqrt{x^4+1} + \sqrt{y^4+1} + \sqrt{z^4+1}}{x^2+y^2+z^2} \leq \sqrt{2}.$$

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16. Prove that in any triangle ABC the following relationship holds:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq \sqrt[6]{4RF}\left(\sqrt[4]{ab} + \sqrt[4]{bc} + \sqrt[4]{ca}\right).$$

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17. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\left(a^3(a+1)+b^3(b+1)+c^3(c+1)\right)\cdot\frac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)}\geq 48.$$

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18. Prove that if $a, b, c, d \in \mathbb{R}^*$ then:

$$(abc-ac-bc-ac)^2\leq 4(1+a^2)(1+b^2)(1+c^2).$$

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19. Let a, b, c be positive real numbers. Prove that:

$$\frac{3a^2+5ab+3b^2}{a^2+ab+b^2}+\frac{3b^2+5bc+3c^2}{b^2+bc+c^2}+\frac{3c^2+5ca+3a^2}{c^2+ca+a^2}\leq 11.$$

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20. Prove that if $n \in \{1,2\}$ then in any triangle ABC the following relationship holds:

$$3^{2-n}\left(\frac{a^{2n+1}}{m_a}+\frac{b^{2n+1}}{m_b}+\frac{c^{2n+1}}{m_c}\right)\geq 2\sqrt{3}(abc)^{\frac{2n}{3}}.$$

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21. Let x, y be positive real numbers with $xy = 3$. Find the minimum value of expression: $\sqrt{x^2+1}+\sqrt{y^2+1}$.

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22. Prove that if $a, b, c \in (0,1)$ are the length's sides in any triangle ABC then:

$$\frac{(s-2)^2+r^2+4rR-1}{(s-1)^2+r^2+4R(r-s)}\geq\frac{3}{\sqrt[3]{(1-a)(1-b)(1-c)}}.$$

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23. Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the maximal value of the expression $A=\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3$.

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24. Prove that in any triangle ABC the following relationship holds:

$$6\sum\frac{a}{2a^2+bc}\leq s\sum\frac{1}{m_a m_b}.$$

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25. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that:

$$\frac{a^{2n+1}}{a^n+(n-1)b^n}+\frac{b^{2n+1}}{b^n+(n-1)c^n}+\frac{c^{2n+1}}{c^n+(n-1)a^n}+\frac{n-1}{n}(a^nb+b^nc+c^na)\geq$$
$$\geq\frac{1}{3^n}$$

for all integers n with $n \geq 1$.

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